SCIENTIFIC PAPERS

Finite groups whose element orders do not exceed twenty*

Seymour LIPSCHUTZ[1] and SHI Wujie (施武杰)[2]

(1. Department of Mathematics, Temple University, Philadelphia PA 19122, USA; 2. Institute of Mathematics, Southwest Normal University, Chongqing 400715, China)

Received December 10, 1998; revised March 14, 1999

Abstract Let $h(\Gamma)$ denote the number of distinct finite groups G such that $\pi_e(G)=\Gamma$. Here Γ is a set of positive integers, and $\pi_e(G)$ is the set of element orders of G. It is proved that $h(\Gamma)\in\{0, 1, 2, \infty\}$ for $\Gamma\subseteq\{1, 2, 3, \cdots, 20\}$.

Keywords: **finite group, element orders, classification theorem.**

1 Introduction and notation

For any set Γ of positive integers, let $h(\Gamma)$ denote the number of distinct groups G such that $\pi_e(G)=\Gamma$, where $\pi_e(G)$ denotes the set of all orders of element in G. The authors in ref. [1] posed the following conjecture: "For any set Γ of positive integers, $h(\Gamma)\in\{0, 1, \infty\}$." This conjecture was included in the 12th edition of *Unsolved Problems in Group Theory* (ref. [2] question 12.84). Recently, Mazurov[3] proved the following results: "If $\Gamma=\pi_e(L_3(5))$, then $h(\Gamma)=2$". Thus he gave a negative answer to the above conjecture. Subsequently, other examples of $h(\Gamma)=2$ have been found[4,5]. On the other hand, the question of $h(\Gamma)=3$ (or any positive integer greater than 2) is still open.

In this paper, we will prove:

Main Theorem. *Let $\Gamma\subseteq\{1, 2, \cdots, 20\}$. Then $h(\Gamma)\in\{0, 1, 2, \infty\}$.*

Furthermore, we classify all finite groups G whose elements orders do not exceed 20. All groups considered in this paper are finite, and all notations are standard (see ref. [6]). In particular, $\pi_e(G)$ denotes the set of all orders of elements in G, and $\pi(G)$ denotes the set of prime divisors of $|G|$.

2 Preliminary lemmas

We need the following lemmas.

Lemma 1[7]. *Let G be a group and N be a minimal normal subgroup of G. Suppose N is an elementary abelian p-group. Then $h(\pi_e(G))=\infty$. In particular, $h(\pi_e(G))=\infty$ when G is a solvable group.*

* Project supported by the National Natural Science Foundation of China (Grant No. 19571066), the State Education Ministry Foundation of China and the NSF of USA (Grant INT. 9600217).

?1994-2018 China Academic Journal Electronic Publishing House. All rights reserved. http://www.cnki.net

A finite simple group G will be called a simple K_n-group if $|\pi(G)| = n$. This generalizes the definition of a simple K_3-group in ref. [8]. Moreover, there are exactly eight simple K_3-groups: A_5, A_6, $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$ and $U_4(2)$[8].

Lemma 2[9]. *Let G be a simple K_4-group. Then G is one of the following groups:*

(i) A_7, A_8, A_9, A_{10};

(ii) M_{11}, M_{12}, J_2;

(iii) $L_2(16)$, $L_2(25)$, $L_2(49)$, $L_2(81)$, $L_3(4)$, $L_3(5)$, $L_3(7)$, $L_3(8)$, $L_3(17)$, $L_4(3)$, $S_4(4)$, $S_4(5)$, $S_4(7)$, $S_4(9)$, $S_6(2)$, $O_8^+(2)$, $G_2(3)$, $U_3(4)$, $U_3(5)$, $U_3(7)$, $U_3(8)$, $U_3(9)$, $U_4(3)$, $U_5(2)$, $Sz(8)$, $Sz(32)$, ${}^3D_4(2)$, ${}^2F_4(2)'$;

(iv) $L_2(r)$, *where r is a prime satisfying the equation*

$$r^2 - 1 = 2^a \cdot 3^b \cdot u^c, \tag{1}$$

where $a \geqslant 1$, $b \geqslant 1$, $c \geqslant 1$, and u is a prime and $u > 3$.

(v) $L_2(2^m)$, *where m satisfies the equations*

$$2^m - 1 = u \text{ and } 2^m + 1 = 3t^b, \tag{2}$$

where u, t are primes and $t > 3$, $b \geqslant 1$.

(vi) $L_2(3^m)$, *where m satisfies the equations*

$$3^m - 1 = u^c \text{ and } 3^m + 1 = 4t$$

or

$$3^m - 1 = 2u \text{ and } 3^m + 1 = 4t^b. \tag{3}$$

where u, t are primes and $b \geqslant 1$, $c \geqslant 1$.

Corollary 1. *Let G be a simple K_4-group and $|G| = 2^a 3^b 5^c 7^d$. Then G is one of the following groups: A_7, A_8, A_9, A_{10}, $L_2(49)$, $L_3(4)$, $U_3(5)$, $U_4(3)$, $S_6(2)$, $S_4(7)$, $O_8^+(2)$, J_2.*

Corollary 2. *Let G be a simple K_4-group and $11 \in \pi(G)$. Then G is one of the following groups: M_{11}, M_{12}, $L_2(q)$, where $q = 11, 23, 32, 243$, $U_5(2)$.*

Group G is called a C_{pp} group if p divides $|G|$ and the centralizer of every nonidentity p-element is a p-group. In ref. [10], the authors determined all C_{pp} simple groups, where p is a prime and $p = 2^a 3^b + 1$, $a \geqslant 0$, $b \geqslant 0$. From the main theorem of ref. [10], we obtain the following results.

Lemma 3. *Let G be a $C_{13,13}$ simple group. Then G is one of the following groups:*

(i) A_{13}, A_{14}, A_{15}, Suz, Fi_{22},

?1994-2018 China Academic Journal Electronic Publishing House. All rights reserved. http://www.cnki.net

(ii) $L_2(q)$, *where* $q = 25, 27, 13^m, 2.3^m - 1$ (*when it is a prime*),

(iii) $L_3(3)$, $L_4(3)$, $O_7(3)$, $S_4(5)$, $S_6(3)$, $O_8^+(3)$, $G_2(3)$, $G_2(4)$, $F_4(2)$, $U_3(4)$, $U_3(23)$, $Sz(8)$, ${}^3D_4(2)$, ${}^2E_6(2)$, ${}^2F_4(2)'$.

Lemma 4. *Let G be a* $C_{17,17}$ *simple group. Then G is one of the following groups:*

(i) A_{17}, A_{18}, A_{19}, J_3, He, Fi_{23}, Fi_{24}',

(ii) $L_2(q)$, *where* $q = 16, 17^m, 2.17^m \pm 1$ (*when it is a prime*),

(iii) $S_4(4)$, $S_8(2)$, $F_4(2)$, $O_8^-(2)$, $O_{10}^-(2)$, ${}^2E_6(2)$.

Lemma 5. *Let G be a* $C_{19,19}$ *simple group. Then G is one of the following groups:*

(i) A_{19}, A_{20}, A_{21}, J_1, J_3, $O'N$, Th, HN,

(ii) $L_2(q)$, *where* $q = 19^m, 2.19^m - 1$ (*when it is a prime*),

(iii) $L_3(7)$, $U_3(8)$, $R(27)$, ${}^2E_6(2)$.

3 Main results

From Lemma 1, we may reduce the proof of our Main Theorem to the products of nonabelian simple groups and their automorphism groups. First we prove the following theorem.

Theorem 1. *Let G be a finite simple group. If the element orders do not exceed* 20, *then G is one of the following simple groups:*

(i) A_5, A_6, A_7, A_8, A_9; $L_2(q)$, $q = 7, 8, 11, 13, 16, 17, 19, 25, 27$; $L_3(q)$, $q = 3, 4, 7$, $L_4(3)$,

(ii) $U_3(q)$, $q = 3, 4, 5$, $U_4(q)$, $q = 2, 3$, $U_5(2)$, $U_6(2)$,

(iii) $Sz(8)$, $S_4(4)$, $S_6(2)$, $G_2(3)$, ${}^2F_4(2)'$, $O_8^+(2)$, $O_7(3)$, $O_8^+(3)$,

(iv) M_{11}, M_{12}, M_{22}, J_1, J_2, J_3, HS.

Proof. Since the element orders do not exceed 20, the number $|\pi(G)|$ of primes dividing $|G|$ does not exceed 8. Also, since G is a finite simple group, we have $3 \leqslant |\pi(G)| \leqslant 8$ from the p^aq^b theorem.

Suppose $|\pi(G)| = 3$, and hence G is a K_3-group. By checking the element orders, we obtain that G is one of the following simple groups: A_5, A_6, $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$, $U_4(2)$.

Suppose $|\pi(G)| = 4$ and $|G| = 2^a3^b5^c7^d$. By Corollary 1 and checking the element orders, we obtain that G is one of the following simple groups: A_7, A_8, A_9, $L_3(4)$, $U_3(5)$, $U_4(3)$,

?1994-2018 China Academic Journal Electronic Publishing House. All rights reserved. http://www.cnki.net

$S_6(2)$, $O_8^+(2)$, J_2.

Suppose $|\pi(G)| = 4$ and $11 \in \pi(G)$. By Corollary 2 and checking the element orders, we obtain that G is one of the following simple groups: M_{11}, M_{12}, $L_2(11)$, $U_5(2)$.

Suppose $|\pi(G)| = 5$ and $|G| = 2^a3^b5^c7^d11^e$. From ref. [11] and checking their element orders, G is one of the following simple groups: M_{22}, HS, $U_6(2)$.

Excluding the above cases, we have $\pi(G) \cap \{13, 17, 19\} \neq \phi$. Since the element orders do not exceed 20, we now get that G is a C_{pp} simple group where $p = 13$, 17, or 19. Suppose G is a $C_{13,13}$ simple group. By Lemma 3 and checking the element orders, we obtain that G is one of the following groups: $L_2(q)$, $q = 12, 25, 27$, $L_3(3)$, $L_4(3)$, $O_7(3)$, $O_8^+(3)$, $G_2(3)$, $U_3(4)$, $Sz(8)$, ${}^2F_4(2)'$.

Suppose G is a $C_{17,17}$ simple group. By Lemma 4 and checking the element orders, we obtain that G is one of the following groups: J_3, $L_2(16)$, $L_2(17)$, $S_4(4)$.

Suppose G is a $C_{19,19}$ simple group. By Lemma 5 and checking the element orders, we obtain that G is one of the following groups: J_1, J_3, $L_2(19)$, $L_3(7)$.

The aforementioned cases include all possibilities. Thus the theorem is proved.

Recall that a group G is said to be almost simple if $M \leqslant G \leqslant \mathrm{Aut}(M)$ where M is a simple group. For the convenience of our discussion, we list the almost simple groups G with $\pi_e(G) \subseteq \{1, 2, \cdots, 20\}$ in table 1.

For the nonsolvable groups in table 1, we have the following results.

Lemma 6[1,12—21]. *Suppose G is one of the following groups: A_5, $A_6.2_3$, A_7, S_7, A_8, A_9, S_9, $L_2(q)$, $q = 7, 8, 11, 13, 16, 17, 19, 25, 27$, $L_3(4)$, $U_3(4)$, $U_4(3)$, $U_6(2)$, $Sz(8)$, M_{11}, M_{12}, M_{22}, J_1, J_3, HS. Then $h(\pi_e(G)) = 1$.*

Lemma 7[5]. *Suppose G is one of the following groups: $S_6(2)$, $O_8^+(2)$, $O_7(3)$, $O_8^+(3)$. Then $h(\pi_e(G)) = 2$.*

Lemma 8[1,12]. *Suppose G is one of the following groups: S_5, A_6, S_6, S_8, J_2. Then $h(\pi_e(G)) = \infty$.*

Lemma 9[22]. *Suppose $\Gamma \subseteq \{1, 2, \cdots, 12\}$. Then $h(\Gamma) \in \{0, 1, \infty\}$.*

The prime graph of G is a graph with the property that its vertex set is the set of primes dividing $|G|$ and two distinct primes p and q are joined by an edge only if there exists an element of order pq in G. Let π_1 be the connected component containing 2 if $|G|$ is even.

Theorem 2. *Suppose G is of one of the following groups: $L_3(7)$, $L_4(3)$, $G_2(3)$, ${}^2F_4(2)'$. Then $h(\pi_e(G)) = 1$. That is, these simple groups can be characterized by their sets of element orders.*

?1994-2018 China Academic Journal Electronic Publishing House. All rights reserved. http://www.cnki.net

Proof. Let H be a group and suppose $\pi_e(H) = \pi_e(G)$. The proof that $H \cong G$ is divided into the following steps.

(i) H is nonsolvable.

First, we assume $G = L_3(7)$. Then $\pi_e(H) = \pi_e(G) = \{1, 2, 3, 4, 6, 7, 8, 14, 16, 19\}$. Suppose H were solvable. Then there would exist a $\{3, 7, 19\}$ - Hall subgroup L of H. Now 3.7, 3.19, and 7.19 do not belong to $\pi_e(H)$, and so all elements of L would be of prime power order. This contradicts the result of Higman[23], and hence H is nonsolvable.

Suppose G is $L_4(3)$, $G_2(3)$, or ${}^2F_4(2)'$, $\pi_e(H) = \pi_e(G)$, and H is solvable. Then there would exist a $\{3, 5, 13\}$, $\{3, 7, 13\}$ or $\{3, 5, 13\}$ - Hall subgroup, respectively. Again, we get a contradiction for all cases, and hence H is nonsolvable.

(ii) H is neither Frobenius nor 2-Frobenius.

First, we assume that H is a Frobenius group with complement C. From Theorem 18.6 in ref. [24] C has a normal subgroup C_0 of index $\leqslant 2$ such that $C_0 \cong SL(2, 5) \times Z$, where every Sylow subgroup of Z is cyclic and $\pi(Z) \cap \{2, 3, 5\} = \phi$. Since $\pi_e(SL(2, 5)) = \{1, 2, \cdots, 6, 10\}$, we get a contradiction by comparing their element orders. The case when H is a 2-Frobenius is similar.

(iii) There exists a normal series $1 \leqslant N \leqslant H_1 \leqslant H$ such that: N and H/H_1 are π_1-groups, and $H^* = H_1/N$ is simple.

This follows directly from the results of Gruenberg and Kegel (see ref. [25]). Furthermore, H^* has at least two components when G is $L_3(7)$, $L_4(3)$, or ${}^2F_4(2)'$, and three components when G is $G_2(3)$.

(iv) $H^* = H_1/N \cong G$.

There are four cases, according as G is $L_3(7)$, $L_4(3)$, $G_2(3)$, or ${}^2F_4(2)'$ respectively.

Case 1. $G = L_3(7)$.

We have $\pi_e(H) = \pi_e(G) = \pi_e(L_3(7)) = \{1, 2, 3, 4, 6, 7, 8, 14, 16, 19\}$ and $\pi_e(H^*) \subseteq \pi_e(H)$. Hence H^* is a $C_{19,19}$ simple group whose element orders do not exceed 19. From table 1, we conclude that H^* can only be $L_2(19)$, $L_3(7)$, J_1, or J_3. However, $L_2(19)$, J_1, and J_3 all contain an element of order 10, and $10 \notin \pi_e(H^*)$. Thus $H^* = L_3(7) = G$.

Case 2. $G = L_4(3)$.

We have $\pi_e(H) = \pi_e(G) = \pi_e(L_4(3)) = \{1, 2, \cdots, 6, 8, 9, 10, 12, 13, 20\}$ and $\pi_e(H^*) \subseteq \pi_e(H)$. Hence H^* is a $C_{13,13}$ simple group whose element orders do not exceed 20. From table 1 and comparing element orders, we conclude that H^* can only be $L_2(25)$, $L_3(3)$ or $L_4(3)$.

?1994-2018 China Academic Journal Electronic Publishing House. All rights reserved. http://www.cnki.net

Table 1 Almost simple groups $M \leqslant G \leqslant \mathrm{Aut}(M)$ with $\pi_e(G) \subseteq \{1, 2, \cdots, 20\}$

M	G	$\pi_e(G)$	M	G	$\pi_e(G)$
A_5	M	1,2,3,5	$L_3(7)$	M	1,2,3,4,6,7,8,14,16,19
	$M.2$	1,2,⋯,6	$L_4(3)$	M	1,2,⋯,6,8,9,10,12,13,20
A_6	M	1,2,⋯,5		$M.2_2$	1,2,⋯,6,8,9,10,12,13,18,20
	$M.2_1$	1,2,⋯,6	$U_3(3)$	M	1,2,3,4,6,7,8,12
	$M.2_2$	1,2,⋯,5,8,10		$M.2$	1,2,3,4,6,7,8,12
	$M.2_3$	1,2,⋯,5,8	$U_3(4)$	M	1,2,⋯,5,10,13,15
	$M.2^2$	1,2,⋯,6,8,10		$M.2$	1,2,⋯,6,8,10,13,15
A_7	M	1,2,⋯,7		$M.4$	1,2,⋯,6,8,10,12,13,15,16
	$M.2$	1,2,⋯,7,10,12	$U_3(5)$	M	1,2,⋯,8,10
A_8	M	1,2,⋯,7,15		$M.2$	1,2,⋯,8,10,12,20
	$M.2$	1,2,⋯,8,10,12,15	$U_4(2)$	M	1,2,⋯,6,9,12
A_9	M	1,2,⋯,7,9,10,12,15		$M.2$	1,2,⋯,6,8,9,10,12
	$M.2$	1,2,⋯,10,12,14,15,20	$U_4(3)$	M	1,2,⋯,9,12
$L_2(7)$	M	1,2,3,4,7		$M.2_1$	1,2,⋯,10,12,14
	$M.2$	1,2,3,4,6,7,8		$M.2_2$	1,2,⋯,10,12,18
$L_2(8)$	M	1,2,3,7,9		$M(2^2)_{122}$	1,2,⋯,10,12,14,18
	$M.3$	1,2,3,6,7,9	$U_5(2)$	M	1,2,⋯,6,8,9,11,12,15,18
$L_2(11)$	M	1,2,3,5,6,11	$U_6(2)$	M	1,2,⋯,12,15,18
	$M.2$	1,2,⋯,6,10,11,12	$Sz(8)$	M	1,2,4,5,7,13
$L_2(13)$	M	1,2,3,6,7,13		$M.3$	1,2,⋯,7,12,13,15
	$M.2$	1,2,3,4,6,7,12,13,14	$S_4(4)$	M	1,2,⋯,6,10,15,17
$L_2(16)$	M	1,2,3,5,15,17		$M.2$	1,2,⋯,6,8,10,12,15,17
	$M.2$	1,2,⋯,6,10,15,17		$M.4$	1,2,⋯,6,8,10,12,15,16,17,20
	$M.4$	1,2,⋯,6,8,10,12,15,17	$S_6(2)$	M	1,2,⋯,10,12,15
$L_2(17)$	M	1,2,3,4,8,9,17	$G_2(3)$	M	1,2,3,4,6,7,8,9,12,13
	$M.2$	1,2,3,4,6,8,9,16,17,18		$M.2$	1,2,3,4,6,7,8,9,12,13,14,18
$L_2(19)$	M	1,2,3,5,9,10,19	${}^2F_4(2)'$	M	1,2,⋯,6,8,10,12,13,16
	$M.2$	1,2,⋯,6,9,10,18,19,20		$M.2$	1,2,⋯,6,8,10,12,13,16,20
$L_2(25)$	M	1,2,⋯,6,12,13	$O_8^+(2)$	M	1,2,⋯,10,12,15
	$M.2$	1,2,⋯,6,10,12,13	$O_7(3)$	M	1,2,⋯,10,12,13,14,15,18,20
	$M.2_3$	1,2,⋯,6,8,12,13	$O_8^+(3)$	M	1,2,⋯,10,12,13,14,15,18,20
$L_2(27)$	M	1,2,3,7,13,14	M_{11}	M	1,2,⋯,6,8,11
	M. 3	1, 2, 3, 6, 7, 9, 13, 14	M_{12}	M	1,2,⋯,6,8,10,11
$L_3(3)$	M	1,2,3,4,6,8,13		$M.2$	1,2,⋯,6,8,10,11,12,
	$M.2$	1,2,3,4,6,8,12,13	M_{22}	M	1,2,⋯,8,11
$L_3(4)$	M	1,2,3,4,5,7		$M.2$	1,2,⋯,8,10,11,12,14
	$M.2_1$	1,2,⋯,8	J_1	M	1,2,3,5,6,7,10,11,15,19
	$M.2_2$	1,2,⋯,8,14	J_2	M	1,2,⋯,8,10,12,15
	$M.2_3$	1,2,⋯,8,10	J_3	M	1,2,⋯,6,8,9,10,12,15,17,19
	$M.2^2$	1,2,⋯,8,10,14	HS	M	1,2,⋯,8,10,11,12,15,20

?1994-2018 China Academic Journal Electronic Publishing House. All rights reserved. http://www.cnki.net

Suppose $H^* = L_2(25)$. Since $9 \notin \pi_e(\mathrm{Aut}(L_2(25))$, we have $3 \in \pi(N)$. We do not lose any generality if we assume N is a 3 group. Consider the inverse image of the Sylow 5 normalizer $N_{H^*}(P_5^*)$ of H^* in H. From Lemma 1 in ref. [15] we get $36 \in \pi_e(H)$, which is a contradiction. Suppose $H^* = L_3(3)$. Since $5 \notin \pi_e(\mathrm{Aut}(L_3(3)))$, we have $5 \in \pi(H)$. We similarly get a contradiction by considering the inverse image of the Sylow 13 normalizer $N_H^*(P_{13}^*)$ of H^* in H. Thus H^* can only be $L_4(3)$.

Case 3. $G = G_2(3)$.

The discussion is similar to Case 2. For the case of $H^* = L_3(3)$ and N is a 7 group. Consider the inverse image of the Sylow 13 normalizer $N_{H^*}(P_{13}^*)$ of H^* in H. From Lemma 6 in ref. [20] we get $21 \in \pi_e(H)$, which is a contradiction. Thus H^* can only be $G_2(3)$.

Case 4. $G = {}^2F_4(2)'$.

We have $\pi_e(H) = \pi_e(G) = \{1, 2, \cdots, 6, 8, 10, 12, 13, 16\}$ and $\pi_e(H^*) \subseteq \pi_e(H)$. Hence H^* is a $C_{13,13}$ simple group whose element orders do not exceed 16. From table 1 and comparing element orders, we conclude that H^* can only be $L_2(13)$, $L_2(25)$, $L_3(3)$ or ${}^2F_4(2)'$. The other discussion is similar to the above cases and we get H^* can only be ${}^2F_4(2)'$.

We have proved $H^* \cong G$, as claimed.

(v) we have $N = 1$.

There are four cases, according as G is $L_3(7)$, $L_4(3)$, $G_2(3)$, or ${}^2F_4(2)'$ respectively.

Case 1. $G = L_3(7)$.

We have $H^* \cong G = L_3(7)$ and $\{3^2, 3.7, 7^2\} \cap \pi_e(L_3(7)) = \phi$. We first claim that N can only be a 2 group. If not, assume N to be a 3 group, then a Sylow 7-subgroup P_7 of H acts fixed-point-freely on N. By Theorem 10.3 in ref. [8] and the fact that $21 \notin \pi_e(H)$, we can conclude that P_7 is cyclic. This implies that $7^3 \in \pi_e(H)$, which is a contradiction. Now suppose $N \neq 1$ is a 2 group. We utilize the Brauer character tables, ref. [26, p. 118], the modular characters of $L_3(7)$ (mod 2) are given. Suppose $x \in H^*$ has order 19, and set $X = \langle x \rangle$. By calculating, we obtain

$$\langle \varphi|_x, 1|_x \rangle = \frac{1}{|X|}\sum_{x \in X}\varphi(x) > 0.$$

Thus by the Corollary of the Main Lemma of ref. [1], $38 \in \pi_e(H)$, which is a contradiction. Thus $N = 1$.

Case 2. $G = L_4(3)$, $G = G_2(3)$ or $G = {}^2F_4(2)'$.

For these cases the discussion is similar. The modular characters of $L_4(3)$ (mod 2) (resp. $L_4(3)$(mod 3)) are given in ref. [26, p.165—166]. The modular characters of $G_2(3)$(mod 2)

?1994-2018 China Academic Journal Electronic Publishing House. All rights reserved. http://www.cnki.net

(resp. $G_2(3)$ (mod 3)) are given in ref. [26, p.140—141]. And the modular characters of ${}^2F_4(2)'$(mod 2) (resp. ${}^2F_4(2)'$(mod 3) and ${}^2F_4(2)'$ (mod 5)) are given in ref. [26, p.188—190]. From the same calculation in Case 1, we get a contradiction. Thus $N = 1$.

In particular, we have proved $H_1 \cong G$.

(vi) Now we prove $H \cong G$.

It suffices to show $H = H_1$. Now $H/C_H(H_1)$ is isomorphic to a subgroup of $\mathrm{Aut}(H_1)$, and $C_H(H_1) = 1$ ($H_1 \cong G$ has at least two components). We have $\mathrm{Aut}(H_1) \geqslant H \geqslant H_1$. Since $H_1 \cong G$, we know $\mathrm{Out}(G)$. (use table 1, and the fact that some subgroups of $\mathrm{Aut}(G)$ are not listed since their element orders exceed 20.) Checking the orders of elements of $\mathrm{Aut}(G)$ one by one, we get $H = H_1 \cong G$. Thus the theorem is proved.

Theorem 3. *Let* $\Gamma = \{1, 2, \cdots, 6, 10, 15, 17\}$. *Then* $\pi_e(S_4(4)) = \pi_e(L_2(16).2) = \Gamma$, *and* $h(\Gamma) = \infty$.

Proof. The first conclusion follows directly from table 1. Let $\Gamma = \{1, 2, \cdots, 6, 10, 15, 17\}$. The group $2^8 : L_2(16)$ which is contained in $O_{10}^-(2)$ has the element orders set Γ. Since the minimal normal subgroup is elementary abelian, we have $h(\Gamma) = \infty$.

Theorem 4. *Let* $\Gamma = \{1, 2, \cdots, 6, 8, 9, 11, 12, 15, 18\}$. *Then* $\pi_e(U_5(2)) = \Gamma$, *and* $h(\Gamma) \in \{1, \infty\}$.

Proof. The first conclusion follows directly from table 1. Now we assume $h(\Gamma) \neq \infty$ and show $h(\Gamma) = 1$.

Let G be a group such that $\pi_e(G) = \Gamma$. Since $h(\pi_e(G)) \neq \infty$, Lemma 1 implies that the minimal normal subgroup N of G is not elementary abelian. Thus we may assume that G is nonsolvable. Also, since $\pi_e(N) \subseteq \pi_e(G)$, N is a nonabelian simple group and $11 \in \pi(N)$ (We note G has two components and $11 \notin \pi_1$), using $\pi_e(N) \subseteq \pi_e(G)$ and checking the simple groups listed in table 1, we have that N is one of the following groups: $L_2(11)$, $U_5(2)$, M_{11} and M_{22}. Now $G/C_G(N) \cong N_G(N)/C_G(N)$ is isomorphic to a subgroup of $\mathrm{Aut}(N)$ and $C_G(N) = 1$; hence $\mathrm{Aut}(N) \geqslant G \geqslant N$. Checking the element orders of $\mathrm{Aut}(N)$, where N is $L_2(11)$, $U_5(2)$, M_{11}, or M_{22}, we obtain that $G \cong U_5(2)$. That is, if $h(\pi_e(G)) \neq \infty$, then $h(\pi_e(G)) = 1$. Thus the theorem is proved.

Lastly, we prove the Main Theorem given in section 1.

Proof. It is only necessary to prove that $h(\pi_e(G)) \in \{1, 2, \infty\}$ for finite groups G whose element orders do not exceed 20. Moreover, if $h(\pi_e(G)) \neq \infty$, then, by Lemma 1, we may assume that N is the minimal normal subgroup of G and $N = T_1 \times T_2 \times \cdots \times T_k$, where each $T_i \cong T$ and T is a nonabelian simple group.

Since $|\pi(T)| \geqslant 3$, we can conclude, using the element orders condition on G, that $k = 1$ or $k = 2$, that is, $N = T$ or $N = T \times T$ (where $T \cong A_5$ or $T \cong A_6$).

?1994-2018 China Academic Journal Electronic Publishing House. All rights reserved. http://www.cnki.net

Suppose $N = T \times T$. Then $\pi_e(N) = \Gamma_1$ or $\pi_e(N) = \Gamma_2$ where $\Gamma_1 = \{1, 2, 3, 5, 6, 10, 15\}$ and $\Gamma_2 = \{1, 2, \cdots, 6, 10, 12, 15, 20\}$. Thus $C_G(N) = 1$, and using $G = N_G(N)/C_G(N)$ which is isomorphic to a subgroup of $\mathrm{Aut}(N)$, we can conclude that $N \leqslant G \leqslant \mathrm{Aut}(N)$. Since $N \cong A_5 \times A_5$ or $N \cong A_6 \times A_6$, we can conclude , using Theorem 3.3.20 of ref. [27], that $\mathrm{Aut}(N) \cong S_5$ wr Z_2 or $\mathrm{Aut}(N) \cong \mathrm{Aut}(A_6)$ wr Z_2. Checking the subgroups of $\mathrm{Aut}(N)$ satisfying the condition of element orders, we get $G = N$ and $\pi_e(G) = \Gamma_1$ or $\pi_e(G) = \Gamma_2$. From $\pi_e(A_5 \times \Sigma_3) = \{1, 2, 3, 5, 6, 10, 15\}$ and $\pi_e(A_6 \times (2^4 : D_{10})) = \{1, 2, \cdots, 6, 10, 12, 15, 20\}$, we obtain $h(\pi_e(G)) = \infty$ by Lemma 1 when $N = T \times T$.

Suppose $N = T$. Then T is a nonabelian simple group listed in table 1. Since $N \lhd G$ and N is a nonabelian simple group, we get $C_G(N) \lhd G$ and $N \cap C_G(N) = 1$.

Suppose $C_G(N) \neq 1$. Then $C_G(N)$ can only be one of the following cases: a 2-group, a 3-group, a 5-group, A_5, or A_6, and $N \cong A_5$ or A_6 except possibly the 2-group case. Supposing $C_G(N)$ is a p-group ($p = 2, 3$, or 5), then G has a minimal normal subgroup $R \neq 1$ and R is an elementary abelian p-group ($p = 2, 3$, or 5). Thus we have $h(\pi_e(G)) = \infty$. Suppose $C_G(N) = A_5$ (resp. A_6) and $N = A_5$ or A_6 (resp. A_5 or A_6). Then G has a normal subgroup K where $K \cong A_5 \times A_5$ (resp. $A_6 \times A_6$ or $A_5 \times A_6$). These cases are the same as the cases discussed above. On the other hand, suppose $C_G(N) = 1$, since $G \cong N_G(N)/C_G(N)$ is isomorphic to a subgroup of $\mathrm{Aut}(N)$, we can conclude that $N \leqslant G \leqslant \mathrm{Aut}(N)$.

Table 1 lists all possible cases of G for which $\pi_e(G) \subseteq \{1, 2, \cdots, 20\}$. We discuss these cases one by one.

Let G be a group satisfying $\pi_e(G) \subseteq \{1, 2, \cdots, 12\}$. Then, by Lemma 9, we have $h(\pi_e(G)) \in \{1, \infty\}$.

Let G be one of the following groups: A_8, A_9, S_9, $L_2(q)$, $q = 13, 16, 17, 19, 25, 27$, $L_3(7)$, $L_4(3)$, $U_3(4)$, $U_4(3)$, $U_6(2)$, $Sz(8)$, $G_2(3)$, ${}^2F_4(2)'$, J_1, J_3, and HS. Then, by Lemma 6 and Theorem 2, $h(\pi_e(G)) = 1$.

Let G be $S_6(2)$, $O_8^+(2)$, $O_7(3)$ or $O_8^+(3)$. Then, by Lemma 7, $h(\pi_e(G)) = 2$.

Let G be S_8 or J_2. Then, by Lemma 8, $h(\pi_e(G)) = \infty$.

Let G be $S_4(4)$ or $L_2(16).2$. Then, by Theorem 3, $h(\pi_e(G)) = \infty$.

Let G be $U_5(2)$. Then, by Theorem 4, $h(\pi_e(G)) \in \{1, \infty\}$.

Now we prove that $h(\pi_e(G)) \in \{1, 2, \infty\}$ for every other group G listed in table 1. They fall into two classes.

Suppose G is one of the following groups:

$L_2(13).2$, $L_2(16).4$, $L_2(17).2$, $L_2(19).2$, $L_2(25).2_2$, $L_2(25).2_3$, $L_2(27).3$,

?1994-2018 China Academic Journal Electronic Publishing House. All rights reserved. http://www.cnki.net

$L_3(3)$, $L_3(3).2$, $L_4(3).2_2$, $U_3(4).2$, $U_3(4).4$, $U_3(5).2$, $U_4(3).2_2$,

$Sz(8).3$, $S_4(4).2$, $S_4(4).4$, $G_2(3).2$, ${}^2F_4(2)'.2$,

Then, as indicated by table 1, G has two components. From Theorem A of ref.[25], we can conclude that G has a normal series $1 \leqslant N \leqslant G_1 \leqslant G$ such that N and G/G_1 are solvable and G_1/N is simple. Suppose $h(\pi_e(G)) \neq \infty$. Then $N = 1$. Again, since $C_G(G_1) = 1$, we have $G_1 \leqslant G \leqslant \mathrm{Aut}(G)$. Table 1 gives the element orders of these groups. Comparing these number sets one by one, we get $h(\pi_e(G)) \in \{1, 2, \infty\}$. (Note that $\pi_e(L_2(16).4)) = \pi_e(S_4(4).2)$).

Let G be one of the remaining groups in table 1; that is, let G be one of the groups $L_3(4).2_2$, $L_3(4).2^2$, $U_4(3).2_1$, $U_4(3).(2^2)_{122}$ or $M_{22}.2$, H be a group and suppose $\pi_e(H) = \pi_e(G)$; N be the minimal normal subgroup of H, if N is elementary abelian, then $h(\pi_e(H)) = h(\pi_e(G)) = \infty$. Otherwise, N is the direct product of isomorphic nonabelian simple groups, say, $N = T_1 \times T_2 \times \cdots \times T_k$. If $k > 1$, then N can only be $A_5 \times A_5$ or $A_6 \times A_6$ where we use the condition that $\pi_e(G) \subseteq \{1, 2, \cdots, 20\}$. Since $15 \notin \pi_e(G)$, we can use $\pi_e(N) \subseteq \pi_e(G)$ to conclude that N is a nonabelian simple group. From $C_H(N) = 1$, we get $N \leqslant H \leqslant \mathrm{Aut}(N)$. Comparing the element orders, using table 1, we get $H = G$ in each case. That is, $h(\pi_e(G)) \in \{1, \infty\}$, for these groups G. Thus our Main Theorem is proved.

4 Open problems

Since we did not find a number set Γ such that $h(\Gamma) = 3$, we have the following natural open problems:

Problem 1. *Find an integer $k > 2$ and groups G such that $h(\pi_e(G)) = k$.*

Problem 2. *Determine whether or not the finite numbers $h(\Gamma)$ are bounded; that is, determine whether or not there exists a positive integer k such that $h(\Gamma) \subseteq \{1, 2, \cdots, k, \infty\}$, for any set Γ of positive integers. In particular, find k if it exists.*

References

1 Praeger, C.E., Shi, W.J., A characterization of some alternating and symmetric groups, *Comm. Algebra*, 1994, 22(5): 1507.

2 Mazurov, V.D., Khukhro, E.I., *Unsolved Problems in Group Theory, Kourovka Notebook*, Novosibirsk: Novosibirsk Press, 1992.

3 Mazurov, V.D., The set of orders of elements in a finite group, *Algebra and Logic*, 1994, 33(1): 49.

4 Chigira, N., Shi, W.J., More on the set of orders of elements in finite groups, *Northeast Math. J.*, 1996, 12(3): 253.

5 Shi, W.J., Tang, C.Y., A characterization of some orthogonal groups, *Progress in Natural Science*, 1997, 7(2): 155.

6 Conway, J.H., Curtis, R.T., Norton, S.P. et al., *ATLAS of Finite Groups*, Oxford: Clarendon Press, 1985.

7 Shi, W.J., Two unsolved new problems in group theory, *J. Southwest China Normal Univ.* (Natural Sci.)(in Chinese), 1996, 21(Suppl. 1): 6.

8 Gorenstein, D., *Finite Groups*, p.6, New York: Harper and Row, 1968.

9 Shi, W.J., On simple K_4-groups, *Chinese Sci. Bull.* (in Chinese), 1991, 36(7): 1281.

10 Chen, Z.M., Shi, W.J., On C_{pp} simple groups, *J. Southwest China Teachers Univ.* (Natural Sci.) (in Chinese), 1993, 18 (3): 249.

11 Cai, Z.F., On the simple groups of order $2^{a_1}3^{a_2}5^{a_3}7^{a_4}p^{a_5}$, *Chin. Ann. of Math.* (in Chinese), 1995, 16(A)(2): 244.

12 Brandl, R., Shi, W.J., Finite groups whose element order are consecutive integers, *J. Algebra*, 1991, 143: 388.

?1994-2018 China Academic Journal Electronic Publishing House. All rights reserved. http://www.cnki.net

13 Brandl, R., Shi, W.J., The characterization of PSL(2, q) by its element orders, *J. Algebra*, 1994, 163: 109.

14 Li, H.L., Shi, W.J., A characteristic property of M_{12} and *PSU* (6,2), *Acta Math. Sinica* (in Chinese), 1989, 32(6): 758.

15 Mazurov, V.D., Characterization of finite groups by sets of orders of their elements, *Algebra and Logic*, 1997, 36(1): 23.

16 Shi, W.J., A characteristic property of A_8, *Acta Math. Sinica* (New Ser.), 1987, 3(1): 92.

17 Shi, W.J., A characterization of some projective linear groups, *J. Math.* (PRC), 1985, 5(2): 191.

18 Shi, W.J., A characterization of the finite group $U_4(3)$, *Analele Univ. Timisoara*, 1992, 30(2-3): 319.

19 Shi, W.J., A characterization of Suzuki's simple groups, *Proc. Amer. Math. Soc.*, 1992, 114(2): 589.

20 Shi, W.J., The characterization of the sporadic simple groups by their element orders, *Algebra Colloquium*, 1994, 1(2): 159.

21 Zhou, F., A characterization of the special projective unitary group $U_3(4)$, *J. Shanghai Jiaotong Univ.* (in Chinese), 1994, 28(3): 106.

22 Shi, W.J., Finite groups whose element orders do not exceed 12, *J. Southwest China Tearchers Univ.* (Natural Sci.)(in Chinese), 1994, 19(4): 341.

23 Higman, G., Finite groups in which every element has prime power order, *J. London Math. Soc.*, 1957, 32: 335.

24 Passman, D.S., *Permutation Groups*, New York: W.A. Benjamin, 1968.

25 Williams, J.S., Prime graph components of finite groups, *J. Algebra*, 1981, 69: 487.

26 Jansen, C., Lux, K., Parker, P. et al., *An Atalas of Brauer Characters*, Oxford: Clarendon Press, 1995.

27 Robinson, D.J.S., *A Course in the Theory of Groups*, New York: Springer-Verlag, New York Inc., 1982.

?1994-2018 China Academic Journal Electronic Publishing House. All rights reserved. http://www.cnki.net